

The Theory of Bonds: A New Method for the Analysis of Linkages

Gábor Hegedüs*, Josef Schicho[†] and Hans-Peter Schröcker[‡]

March 8, 2013

In this paper we introduce a new technique, based on dual quaternions, for the analysis of closed linkages with revolute joints: the theory of bonds. The bond structure comprises a lot of information on closed revolute chains with a one-parametric mobility. We demonstrate the usefulness of bond theory by giving a new and transparent proof for the well-known classification of overconstrained 5R linkages.

Keywords: Dual quaternions, bond theory, overconstrained revolute chain, overconstrained 5R chain.

MSC 2010: 70B15, 51J15, 14H50

Introduction

In this paper we rigorously develop the theory of bonds [9], a tool for the analysis of closed linkages with revolute joints. The configuration curve of such a linkage can be described by algebraic equations. Intuitively, bonds are points in the configuration curve with complex coefficients where something degenerate happens. For a typical bond of a closed n R chain, there are exactly two joints with degenerate rotation angles (see Theorem 8 and the subsequent remark). In this way, the bond “connects” the two links. It is remarkable that a lot of information on the linkage can be extracted from this combinatorial behavior of the bonds.

In order to describe the forward kinematic map from the configuration curve into the group of Euclidean displacements, we use the language of dual quaternions (see also

*Kecskemét College, Faculty of Mechanical Engineering and Automation, <http://www.gamf.hu/portal2/kecskem-et-college>, hegedus.gabor@gamf.kefo.hu

[†]Johann Radon Institute for Computational and Applied Mathematics, Austrian Academy of Sciences (RICAM), <http://www.risc.jku.at/people/jschicho/>, josef.schicho@oeaw.ac.at

[‡]University Innsbruck, Unit Geometry and CAD, <http://geometrie.uibk.ac.at/schroecker/>, hans-peter.schroecker@uibk.ac.at

[3, 6, 10, 12]). For any pair of links, the set of possible relative poses is a curve on the Study quadric in the projective space \mathbb{P}^7 . In Theorem 17 we compute the degrees of these curves by the combinatorial behavior of the bonds.

Theorem 5 is interesting in its own right. It relates the geometry of three consecutive revolute axes with the dimension of a certain linear subspace of the 8-dimensional vector space \mathbb{DH} of dual quaternions. In general, this subspace is equal to \mathbb{DH} , but, it may also be of dimension 4 or 6 for particular positions of the three lines. More precisely, the dimension is 4 if and only if the three lines are parallel or meet in a common point, and it is 6 if and only if the lines appear as revolute axes in a 4R Bennett linkage. We show that a bond of certain type appears if and only if the dimension of the corresponding linear subspace is less than 8.

Section 3 features a rigorous definition of bonds and connection numbers. We visualize the latter in bond diagrams and show how to read off linkage properties. As an application of bond theory, we give a new proof of Karger's classification theorem for overconstrained closed 5R-linkages [11] in Section 4. In contrast to Karger's original proof, it does not require the aid of a computer algebra system.

We announced simplified versions of the results of this paper in [9] without proofs. Supplementary material to this article can be found on the accompanying web-site <http://geometrie.uibk.ac.at/schroecker/bonds/>.

1 Dual quaternions

In this section, we recall the well-known and classical description of the group of Euclidean displacements by dual quaternions; it is almost identical to [8, Section 2]. We just include it here to make this paper more self-contained. More complete references are [3, 6, 10, 12].

We denote by SE_3 the group of direct Euclidean displacements, i.e., the group of maps from \mathbb{R}^3 to itself that preserve distances and orientation. It is well-known that SE_3 is a semidirect product of the translation subgroup and the orthogonal group SO_3 , which may be identified with the stabilizer of a single point.

We denote by $\mathbb{D} := \mathbb{R} + \epsilon\mathbb{R}$ the ring of dual numbers, with multiplication defined by $\epsilon^2 = 0$. The algebra \mathbb{H} is the non-commutative algebra of quaternions, and \mathbb{DH} is the algebra of quaternions with coefficients in \mathbb{D} . Every dual quaternion has a primal and a dual part (both quaternions in \mathbb{H}), a scalar part in \mathbb{D} and a vectorial part in \mathbb{D}^3 . The conjugate dual quaternion \bar{h} of h is obtained by multiplying the vectorial part of h by -1 . The dual numbers $N(h) = h\bar{h}$ and $h + \bar{h}$ are called the *norm* and *trace* of h , respectively.

By projectivizing \mathbb{DH} as a real 8-dimensional vectorspace, we obtain \mathbb{P}^7 . The condition that $N(h)$ is strictly real, i.e. its dual part is zero, is a homogeneous quadratic equation. Its zero set, denoted by S , is called the Study quadric. The linear 3-space represented by all dual quaternions with zero primal part is denoted by E . It is contained in the Study quadric. The complement $S - E$ can be identified with SE_3 . The primal part describes SO_3 . Translations correspond to dual quaternions with primal part ± 1 and strictly vectorial dual part. More precisely, the group isomorphism is given by sending

$h = p + \epsilon q$ to the map

$$\mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad v \mapsto \frac{pv\bar{p} + p\bar{q} - q\bar{p}}{p\bar{p}}.$$

(see [3, p. 48] or [10, Section 2.4]).

A nonzero dual quaternion represents a rotation if and only if its norm and trace are strictly real and its primal vectorial part is nonzero. It represents a translation if and only if its norm and trace are strictly real and its primal vectorial part is zero. The 1-parameter rotation subgroups with fixed axis and the 1-parameter translation subgroups with fixed direction can be geometrically characterized as the lines on S through the identity element 1. Among them, translations are those lines that meet the exceptional 3-plane E .

2 Linkages

In this section, we introduce some terminology on linkages, like coupler curves and coupling spaces (relative motions between links, described in terms of dual quaternions and linear spans of these curves), and prove a useful theorem about the dimension of coupling spaces.

We describe an open chain of $n > 0$ revolute joints by a sequence $L = (h_1, \dots, h_n)$ of unit dual quaternions h_1, \dots, h_n of zero scalar part. Algebraically, this means that $h_i \bar{h}_i = -h_i^2 = 1$. Geometrically, we represent a revolute joint by a half-turn (a rotation by the angle π). The group parametrized by $(t - h_i)_{t \in \mathbb{P}^1}$ – the parameter t determines the rotation angle – is the group of the $(i + 1)$ -th link relative to the i -th link. The position of the last link with respect to the first link is then given by a product $(t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n)$, with $t_1, \dots, t_n \in \mathbb{P}^1$. For a closed chain, we have the closure condition

$$(t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n) \in \mathbb{R} \setminus \{0\}. \quad (1)$$

We view closed chains as cyclic sequences $L = (h_1, \dots, h_n)$ and we reflect this in the notational convention $h_{kn+i} := h_i$ for $k \in \mathbb{Z}$.

Definition 1. For a closed chain of revolute joints as described above, the set K of all n -tuples $(t_1, \dots, t_n) \in (\mathbb{P}^1)^n$ fulfilling (1) is called the chain’s *configuration curve*.

The dimension of the configuration curve is called the *degree of freedom* or the *mobility* of the linkage. In this paper we consider linkages of mobility one. This already implies $4 \leq n \leq 7$. For $n = 4$, we obtain planar, spherical or spatial four bar linkages. The latter are usually referred to as Bennett linkages. In general, closed chains of $n < 7$ revolute joints are rigid. Thus, our results in this paper refer to planar and spherical four bar linkages, to linkages of paradoxical mobility with less than seven joints, and to linkages with seven joints and one degree of freedom.

A linkage is a set of links, a set of joints, and a relation between them, which we call “attachment”. Any link has at least one attached joint, and any joint has at least two attached links. If two joints are attached to two links, then either the two joints or the two links are equal. The link diagram is a linear hypergraph whose vertices are the links and

whose hyperedges are the joints; dually, the joint diagram is a linear hypergraph whose vertices are the joints and whose hyperedges are the links. In both cases, hyperedges are needed because a link can have more than two attached joints and a joint can be attached to more than two links. In this paper we will mostly be concerned with open and closed chains with revolute joints. Here the two hypergraphs are just simple graphs, consisting of a path or cycle. Nonetheless, it should be kept in mind that the theory we develop can also be applied to cycles in general linkages.

To each revolute joint we attach its axis of rotation (a line in \mathbb{R}^3). It can be represented by the same dual quaternion h_i as the joint. This is almost the same as the representation of lines by normalized Plücker coordinates which are composed of primal part and negative dual part. The line determines h_i up to multiplication with -1 . A configuration of a linkage consists of the specification of suitable revolute angles for each pair of links joined by a joint. This angle corresponds to a rotation of the form $t_i - h_i$, $t_i \in \mathbb{R}$, or to the identity 1 for $t_i = \infty$.

Let $L = (h_1, \dots, h_n)$ be a closed nR chain with mobility one. We denote the links by o_1, \dots, o_n , and use the convention that o_i is the link with joint axes h_i, h_{i+1} for $i = 1, \dots, n$. We use $[n]$ as shorthand notation for the set $\{1, \dots, n\}$. For $i, j \in [n]$, we define the polynomial

$$F_{i,j} = (t_{i+1} - h_{i+1})(t_{i+2} - h_{i+2}) \cdots (t_j - h_j) \in \mathbb{DH}[t_i, \dots, t_j] \subset \mathbb{DH}[t_1, \dots, t_n], \quad (2)$$

and the map

$$f_{i,j}: K \rightarrow \text{SE}_3, \\ (t_1, \dots, t_n) = \tau \mapsto \begin{cases} F_{i,j}(\tau) & \text{if } F_{i,j}(\tau) \neq 0, \\ \lim_{\tau' \rightarrow \tau} F_{i,j}(\tau') & \text{else.} \end{cases} \quad (3)$$

The distinction between the polynomial $F_{i,j}$ and the map $f_{i,j}$ is necessary because $F_{i,j}(\tau)$ may vanish at isolated points $\tau \in K$ (see Corollary 10 and Example 5), that is, the evaluation of $F_{i,j}$ at points $\tau \in K$ does not give a well-defined map into \mathbb{P}^7 . On the other hand, the map $f_{i,j}$ is well-defined for all regular points $\tau \in K$. (Thus, it should actually be defined on the normalization $N(K)$ of K , compare Section 3.1.)

Because of the closure condition (1), we also have

$$f_{i,j}(\tau) = \begin{cases} \tilde{F}_{i,j}(\tau) & \text{if } \tilde{F}_{i,j}(\tau) \neq 0, \\ \lim_{\tau' \rightarrow \tau} \tilde{F}_{i,j}(\tau') & \text{else,} \end{cases} \quad (4)$$

where

$$\tilde{F}_{i,j} = \prod_{k=1}^{n+i-j} (t_{i-k+1} - h_{i-k+1}) = (t_i + h_i)(t_{i-1} + h_{i-1}) \cdots (t_{j+1} + h_{j+1}).$$

Definition 2. The map $f_{i,j}$ defined in (3) is called the *coupler map*. Its image is the *coupler curve* $C_{i,j}$.

The coupler curve $C_{i,j}$ describes the motion of link o_j relative to link o_i . This is the reason for the seemingly strange index convention in the definition of the coupler map $f_{i,j}$.

Definition 3. For a sequence h_i, h_{i+1}, \dots, h_j of consecutive joints, we define the *coupling space* $L_{i,i+1,\dots,j}$ as the linear subspace of \mathbb{R}^8 generated by all products $h_{k_1} \cdots h_{k_s}$, $i \leq k_1 < \dots < k_s \leq j$. (Here, we view dual quaternions as real vectors of dimension eight.) The empty product is allowed, its value is 1. The *coupling dimension* $l_{i,i+1,\dots,j}$ is the dimension of $L_{i,i+1,\dots,j}$ and the *coupling variety* $X_{i,i+1,\dots,j} \subset \mathbb{P}^7$ is the set of all products $(t_i - h_i) \dots (t_j - h_j)$ with $t_k \in \mathbb{P}^1$ for $k = i, \dots, j$ or, more precisely, the set of all equivalence classes of these products in the projective space.

The coupling variety is a subset of the projectivization of the coupling space. The relation between the coupler curve and the coupling variety is described by the “coupler equality” $C_{i,j} = X_{i+1,\dots,j} \cap X_{i-1,\dots,-n+j}$. We also recall the nomenclature of [7, 8]: Two rotation quaternions with the same axes are called *compatible*. Moreover, two or more lines are called *concurrent* if they are all parallel or intersect in a common point.

We now prove a theorem that relates the introduced concepts to the axis geometry of the linkage. We will use it later to show that bonds have a geometric meaning but it has aspects (for example Theorem 5.d) which are interesting in its own right.

Lemma 4. *The triple $(L_1, +, \cdot)$ is a field and isomorphic to \mathbb{C} .*

Proof. The set $L_1 = \{a + bh_1 \mid a, b \in \mathbb{R}\}$ is closed under addition. Since quaternions in L_1 describe rotations about one fixed axis it is also closed under multiplication and inversion. This already implies that L_1 is a field. Because of $h_1^2 = -1$, L_1 is isomorphic to \mathbb{C} . \square

Theorem 5. *If h_1, h_2, \dots, h_n are rotation quaternions such that h_i and h_{i+1} are not compatible for $i = 1, \dots, n-1$, the following statements hold true:*

- a) *All coupling dimensions $l_{1,\dots,i}$ with $1 \leq i \leq n$ are even.*
- b) *The equation $l_{12} = 4$ always holds. Moreover, $L_{12} \subset S$ if and only if the axes of h_1 and h_2 are concurrent.*
- c) *If $\dim L_{123} = 4$, then the axes of h_1, h_2, h_3 are concurrent.*
- d) *If $\dim L_{123} = 6$, then the axes of h_1, h_2, h_3 satisfy the Bennett conditions: the normal feet of h_1 and h_3 on h_2 coincide and the normal distances $d_{i,i+1}$ and angles $\alpha_{i,i+1}$ between consecutive axes are related by $d_{12}/\sin \alpha_{12} = d_{23}/\sin \alpha_{23}$.*

Proof. a) The coupling space $L_{1,\dots,i}$ is closed under multiplication with L_1 from the left. Hence $L_{1,\dots,i}$ is a vector space over the field L_1 . By Lemma 4, L_1 is isomorphic to \mathbb{C} . Hence, the real dimension of $L_{1,\dots,i}$ is even.

b) is well-known [12, Section 11.2.1].

c) Since both L_{123} and L_{12} have dimension 4, the two vectorspaces are equal and $h_3 \in L_{12}$. Assume, indirectly, that h_1 and h_2 are not concurrent. Then the only rotations in the projectivizations are compatible with h_1 or h_2 . By assumption, h_3 is not compatible with h_2 , hence $h_3 = \pm h_1$. Then L_{123} is closed under multiplication by $h_1 = \pm h_3$ from the left and from the right. On the other hand, no proper subalgebra of \mathbb{DH} can contain two skew lines, hence h_1 and h_2 are concurrent.

d) If $h_3 \in L_{12}$, it is either compatible with h_1 or h_2 . The latter is excluded by assumption, the former satisfies the Bennett conditions. Hence, we can assume $h_3 \notin L_{12}$ and the vectors $1, h_1, h_2, h_3, h_1h_2$ are linearly independent. As an L_1 -vectorspace, L_{123} is generated by $1, h_2, h_3, h_2h_3$. Assume that these vectors form a basis of L_{123} . Then $w + xh_2 + yh_3 + zh_2h_3 = 0$ with $w, x, y, z \in L_1$ would imply $w = x = y = z = 0$ so that $l_{123} = 8$. This contradicts our assumption. Hence, there is a non-trivial linear relation

$$x + yh_2 + zh_3 = h_2h_3 \quad (5)$$

with unique $x, y, z \in L_1$. By multiplying (5) from the right with h_3 , we obtain $xh_3 + yh_2h_3 - z = -h_2$. Comparing coefficients with (5) then yields $y^2 = -1$, $z = xy$, and $x = zy$. We may assume, possibly after replacing h_1 by $-h_1$, that $y = -h_1$. Then we can also write $x = a + bh_1$ and $z = b - ah_1$ for some $a, b \in \mathbb{R}$. If $a = 0$, (5) becomes $(h_2 - b)h_3 = h_1(b - h_2)$, there is a rotation around h_2 that transforms h_1 to h_3 and the claim follows. If $a \neq 0$, we set $h'_2 := a^{-1}(h_2 - b)$ (another rotation about the same axis) and find

$$\begin{aligned} a(h_1h'_2 + h_1h_3 + h'_2h_3) &= \\ h_1h_2 - bh_1 + ah_1h_3 + h_2h_3 - bh_3 &= \\ h_1h_2 - bh_1 + ah_1h_3 + a + bh_1 - h_1h_2 + bh_3 - ah_1h_3 - bh_3 &= a. \end{aligned}$$

It follows that $h_4 := -h_1 - h'_2 - h_3$ fulfills the two equations $h_1 + h'_2 = \overline{h_4} + \overline{h_3}$, $h_1h'_2 = \overline{h_4} \overline{h_3}$. Hence, the closure equation $(t - h_1)(t - h_2)(t - h_3)(t - h_4) \in \mathbb{R}$ of Bennett's mechanism is fulfilled (see [7, 8]). \square

3 Bonds

In this article's central section we define bonds and introduce the bond structure (local distance and local joint length). We show how the bond structure can be used to compute the degree of coupler curves and derive some algebraic implications of the theory. Towards the end of this section, we introduce the connection numbers associated to bonds and use them for drawing bond diagrams. From now on, we consider closed revolute chains with incompatible consecutive axes only.

3.1 Definition of bonds

Consider a closed chain $L = (h_1, \dots, h_n)$ of mobility one with configuration curve K . By $K_{\mathbb{C}}$ we denote its Zariski closure, the set of all points in $(\mathbb{P}_{\mathbb{C}}^1)^n$ which satisfy all algebraic equations that are also satisfied by all points of K . Now we set

$$B := \{(t_1, \dots, t_n) \in K_{\mathbb{C}} \mid (t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n) = 0\}. \quad (6)$$

Proposition 6. *We have $\dim(B) = \dim(K) - 1$.*

Proof. The ideal of B is generated by the ideal of K and one additional equation, the primal scalar part of $(t_1 - h_1)(t_2 - h_2) \cdots (t_n - h_n)$. Hence B is a hypersurface in $K_{\mathbb{C}}$ and it follows that $\dim(B) \geq \dim(K) - 1$. If $\dim(B) = \dim(K)$, then there would be a component of $K_{\mathbb{C}}$ that would entirely lie in B . But this is impossible because B has no real points and K is entirely real. \square

The set B is a finite set of conjugate complex points on the configuration curve's Zariski closure. These points are special in the sense that they, by defining condition (6) of bonds, do not correspond to a valid linkage configuration.

In [9], we simply defined a bond as a point of B . But we also remarked that this is only valid in “typical” cases. Here, we adopt a more general point of view. It is conceivable that $K_{\mathbb{C}}$ is singular at a point of B so that more than one bond lies over this point. In order to overcome this technical difficulty, we consider the *normalization* $N(K)$ instead of $K_{\mathbb{C}}$ (see [13] Chapter II.5). The normalization $N(K)$ is a singularity-free curve that serves as parameter range for $K_{\mathbb{C}}$. In other words, there exists a surjection $\nu: N(K) \rightarrow K_{\mathbb{C}}$, the *normalization map*.

Definition 7. Let $N(K)$ be the normalization of the algebraic curve $K_{\mathbb{C}}$, with normalization map $\nu: N(K) \rightarrow K_{\mathbb{C}}$. A point $\beta \in N(K)$ is called a *bond* if $\nu(\beta) \in B$.

We mention that it is usually possible to think of a bond as a point $\beta \in B$. The concept of normalization is only needed if $K_{\mathbb{C}}$ is singular at β – a situation we will not encounter in this paper.

In the following, we denote the standard basis of the dual quaternions \mathbb{DH} by $(1, \mathbf{i}, \mathbf{j}, \mathbf{k}, \epsilon, \epsilon\mathbf{i}, \epsilon\mathbf{j}, \epsilon\mathbf{k})$ and the imaginary unit in the field of complex numbers \mathbb{C} by i . Often, complex numbers are embedded into the quaternions by identifying i with \mathbf{i} . In this paper, we do not do this. *It is crucial to distinguish between the imaginary unit i and the quaternion \mathbf{i} .* We will, for example, encounter expressions like $i - \mathbf{i}$. This is a quaternion with complex coefficients and different from zero.

As a first example, we compute the bonds of a Bennett linkage. (The source code for computing the following examples can be found on the accompanying web-site <http://geometrie.uibk.ac.at/schroecker/bonds/>.)

Example 1 (Bennett linkage). Consider the linkage $L = (h_1, h_2, h_3, h_4)$ with

$$\begin{aligned} h_1 &= \mathbf{i}, \\ h_2 &= 9\epsilon\mathbf{i} + \mathbf{j} - 9\epsilon\mathbf{k}, \\ h_3 &= -(\tfrac{1}{3} + 4\epsilon)\mathbf{i} - (\tfrac{2}{3} - 4\epsilon)\mathbf{j} + (\tfrac{2}{3} + 2\epsilon)\mathbf{k}, \\ h_4 &= (\tfrac{2}{3} + 5\epsilon)\mathbf{i} + (\tfrac{1}{3} + 4\epsilon)\mathbf{j} + (\tfrac{2}{3} - 7\epsilon)\mathbf{k}. \end{aligned}$$

From the closure condition $(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) \in \mathbb{R}$ we can compute the parametrized representation

$$t_1 = t - 1, \quad t_2 = t, \quad t_3 = t - 1, \quad t_4 = -t, \quad t \in \mathbb{P}^1 \tag{7}$$

of the configuration curve. It is, indeed, of dimension one and L is a flexible closed 4R chain. Since it is neither spherical nor planar, it is a Bennett linkage. We have

$$(t - 1 - h_1)(t - h_2)(t - 1 - h_3)(-t - h_4) = -(t^2 + 1)(t^2 - 2t + 2). \quad (8)$$

The bonds can be computed by solving $(t_1 - h_1)(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) = 0$. This means, that we have to find the zeros of (8). They are $t = \pm i$ and $t = 1 \pm i$ so that the bond set B consists of the points

$$(\pm i, 1 \pm i, \pm i, -1 \mp i), \quad (-1 \pm i, \pm i, -1 \pm i, \mp i). \quad (9)$$

We observe that every bond of (9) has two entries equal to i or $-i$. As next theorem shows, this is no coincidence but a typical property of bonds. \diamond

Theorem 8. *For a bond $\beta \in \nu^{-1}(t_1, \dots, t_n)$ there exist indices $i, j \in [n]$, $i < j$, such that $t_i^2 + 1 = t_j^2 + 1 = 0$.*

Proof. Observe at first that for any $k \in [n]$ the equality

$$N(t_k - h_k) = (t_k - h_k)(\overline{t_k - h_k}) = (t_k - h_k)(t_k + h_k) = t_k^2 + 1$$

holds. ($N(h) = h\bar{h}$ is the norm of a dual quaternion.) Taking the norm on both side of the defining condition (6) of bonds, we obtain

$$0 = \prod_{k=1}^n (t_k - h_k) \prod_{k=1}^n (t_{n+1-k} + h_{n+1-k}) = \prod_{k=1}^n (t_k^2 + 1). \quad (10)$$

We conclude that $t_i^2 + 1 = 0$ for at least one index $i \in [n]$ and we assume that i is the minimal index with this property. In order to show existence of a second index $j \in [n]$, $i < j$ with $t_j^2 + 1 = 0$, we successively multiply the bond equation (6) with $t_n + h_n, \dots, t_{i+1} + h_{i+1}$ from the right and with $t_1 + h_1, \dots, t_{i-1} + h_{i-1}$ from the left. The result is

$$0 = \prod_{k=1}^{i-1} (t_k + h_k) \prod_{k=1}^n (t_k - h_k) \prod_{k=1}^{n-i} (t_{n+1-k} + h_{n+1-k}) = (t_i - h_i) \prod_{k \neq i} (t_k^2 + 1).$$

Now the claim follows because $t_i - h_i$ never vanishes. \square

Definition 9. We call a bond $\beta = \nu^{-1}(t_1, \dots, t_n)$ *typical* if there are precisely two indices $i, j \in [n]$, $i < j$ such that $t_i^2 + 1 = t_j^2 + 1 = 0$.

Theorem 8 is important for two reasons. First of all, it gives us necessary conditions that are useful for the actual computation of typical bonds. Secondly, it is a further manifestation of the mentioned discrete properties of bonds: For a typical bond β , the two links h_i, h_j with $t_i^2 + 1 = t_j^2 + 1 = 0$ play a special role. We say that the bond “connects” h_i and h_j . However, this concept requires a more refined elaboration as we also have to take into account non-typical cases and higher connection multiplicities. For this reasons, the precise definition of a connection number between two joints is necessary. This needs more preparation work and will be deferred until Section 3.4.

Corollary 10. For a typical bond $\beta = \nu^{-1}(t_1, \dots, t_n)$ with $t_i^2 + 1 = t_j^2 + 1 = 0$ and $i < j$, the equalities

$$F_{i-1,j}(t_1, \dots, t_n) = F_{j-1,n+i}(t_1, \dots, t_n) = 0 \quad (11)$$

hold.

Proof. Once more, we consider the bond equation (6). We multiply it from the left with $t_1 + h_1, \dots, t_{i-1} + h_{i-1}$ and from the right with $t_{j+1} + h_{j+1}, \dots, t_n + h_n$ to obtain

$$0 = \prod_{k=1}^{i-1} (t_k + h_k) \prod_{k=1}^n (t_k - h_k) \prod_{k=j+1}^n (t_k + h_k) = \prod_{k \notin \{i, \dots, j\}} (t_k^2 + 1) \prod_{k=i}^j (t_k - h_k).$$

Because the first product on the right is different from zero, the second vanishes. The second equality can be seen similarly. \square

The reader is invited to verify Corollary 10 with the data of Example 1.

Before proceeding with our study of bonds, we present two further examples (spherical and planar four-bar linkage) that illustrate special situations that can occur: Different bonds may have the same indices $i < j \in [n]$ such that $t_i^2 + 1 = t_j^2 + 1 = 0$ and, for a given bond, there might exist more than two indices $i < j \in [n]$ with this property.

Example 2 (Spherical four-bar linkage). We consider the spherical four-bar linkage $L = (h_1, h_2, h_3, h_4)$ given by

$$h_1 = \mathbf{i}, \quad h_2 = \mathbf{j}, \quad h_3 = \mathbf{k}, \quad h_4 = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}.$$

The configuration curve admits the parametrization

$$t_1 = \frac{5 - 5t^2 + w}{6t}, \quad t_2 = \frac{-5t^2 - 5 + w}{8t}, \quad t_3 = \frac{25t^2 - 7 - 5w}{24}, \quad t_4 = t \quad (12)$$

where $w = \pm\sqrt{25t^4 - 14t^2 + 25}$. The bonds are

$$(\mp 3\mathbf{i}, \mp \mathbf{i}, -3, \pm \mathbf{i}), \quad (\mp \frac{1}{3}\mathbf{i}, \pm \mathbf{i}, \frac{1}{3}, \pm \mathbf{i}), \quad (\mp \mathbf{i}, -1, \pm \mathbf{i}, \frac{4}{5} \pm \frac{3}{5}\mathbf{i}), \quad (\mp \mathbf{i}, 1, \mp \mathbf{i}, -\frac{4}{5} \pm \frac{3}{5}\mathbf{i}).$$

Thus, we have two pairs of conjugate complex bonds with $t_1^2 + 1 = t_3^2 + 1 = 0$ and two pairs of conjugate complex bonds with $t_2^2 + 1 = t_4^2 + 1 = 0$. \diamond

Example 3 (Planar four-bar linkage). The configuration curve of the planar four-bar linkage given by

$$h_1 = \epsilon \mathbf{i} + \mathbf{k}, \quad h_2 = \epsilon \mathbf{j} + \mathbf{k}, \quad h_3 = \mathbf{k}, \quad h_4 = \epsilon \mathbf{i} + 2\epsilon \mathbf{j} + \mathbf{k}$$

can be parametrized by

$$t_1 = \frac{-t^2 + 2t + 5 - w}{2(t + 3)}, \quad t_2 = \frac{t^2 + 1 + w}{4(2 - t)}, \quad t_3 = \frac{t^2 - 4t + 1 + w}{4(t - 1)}, \quad t_4 = t \quad (13)$$

where $w = \pm\sqrt{t^4 - 8t^3 + 2t^2 + 56t - 47}$. The bonds are

$$(\pm i, -2 \pm i, \mp i, 4 \mp i), \quad (2 \pm i, \mp i, -1 \pm 2i, \pm i), \quad (\pm i, \mp i, \pm i, \mp i).$$

The special thing here is the existence of two non-typical bonds. For them, Corollary 10 cannot be applied. Nonetheless, we observe that

$$\begin{aligned} (t_1 - h_1)(t_2 - h_2)(t_3 - h_3) &= (t_2 - h_2)(t_3 - h_3)(t_4 - h_4) = \\ (t_3 - h_3)(t_4 - h_4)(t_1 - h_1) &= (t_4 - h_4)(t_1 - h_1)(t_2 - h_2) = 0 \end{aligned}$$

holds for $(t_1, t_2, t_3, t_4) = (\pm i, \mp i, \pm i, \mp i)$. \diamond

Remark 11. So far, we silently ignored the possibility of a bond with coordinate ∞ . Actually, no such bonds occur in our examples but this has to be checked carefully. Linkages with “bonds at infinity” do exist.

Example 3 is an indication that the vanishing of coupler maps as stated in Corollary 10 for typical bonds, is a more relevant property than existence of indices $i < j \in [n]$ with $t_i^2 + 1 = t_j^2 + 1 = 0$, as stated by Theorem 8. In the following, we will elaborate this concept in more detail.

3.2 Local distances and joint lengths

Now we are going to define local distances and joint lengths of a linkage. These are algebraic notions related to a single bond. In Section 3.3 we will define (non-local) distances and joint lengths as sum over all local distances and joint lengths, respectively.

The definition of local distances requires the concept of the vanishing order of a function $f: K_{\mathbb{C}} \rightarrow \mathbb{P}^7$ at a bond β . Consider an arbitrary homogeneous quadratic form $F: \mathbb{R}^8 \rightarrow \mathbb{C}$. The image $F(x)$ of a vector $x \in \mathbb{R}^8$ is obtained by plugging the coordinates of x into a homogeneous quadratic polynomial. The function F is not necessarily well-defined on \mathbb{P}^7 since $F(x)$ might be zero, but the vanishing order $\text{ord}_{\beta}(F(f))$ of $F \circ f$ at β is well-defined. In this article, we will use the homogeneous quadratic form Q which maps $x = (x_0, \dots, x_7) \in \mathbb{R}^8$ to $Q(x) = x_0^2 + x_1^2 + x_2^2 + x_3^2$ (the primal part of $N(x)$).

Definition 12. For a bond $\beta \in N(K)$ and a pair (i, j) of links, the *local distance* is defined as $d_{\beta}(i, j) := \frac{1}{2} \text{ord}_{\beta} Q(f_{i,j})$ where $f_{i,j}$ is the coupler map of Definition 2. The *local distance matrix* D_{β} is the matrix with entries $d_{\beta}(i, j)$. The *local joint length* is defined as $b_{\beta}(i) := d_{\beta}(i - 1, i) = \frac{1}{2} \text{ord}_{\beta} Q(t_i - h_i)$.

Remark 13. Definition 12 relates bond theory to a familiar concept of theoretical kinematics. If $d_{\beta}(i, j)$ is positive, the image of the bond β under the coupling map f_{ij} is a point $x = (x_0, \dots, x_7) \in C_{ij}$ such that $x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0$. This equation describes a quadratic cone G whose vertex space is the exceptional 3-plane E . Intersections of the coupler curve with E have been considered before (e.g. in [12, Chapter 11] and [10]), this article shows that it is even more interesting to study the intersection points with G .

Example 4. The local distance matrices for the Bennett linkage of Example 1 are

$$D_{\beta'} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad D_{\beta''} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad (14)$$

with $\beta' = (\pm i, 1 \pm i, \pm i - 1 \mp i)$ and $\beta'' = (-1 \pm i, \pm i, -1 \pm i, \mp i)$. For their computation, we can use the parametrized representation (7). We find, for example,

$$\begin{aligned} F_{1,3}(t) &= (t - h_2)(t - 1 - h_3) = t^2 - t + 5\epsilon + \frac{2}{3} \\ &\quad + (t(\frac{1}{3} - 5\epsilon) + \frac{2}{3} + 5\epsilon)\mathbf{i} + (-t(\frac{1}{3} + 4\epsilon) + 1 - 3\epsilon)\mathbf{j} \\ &\quad + (-t(\frac{2}{3} + 7\epsilon) + \frac{1}{3} - 11\epsilon)\mathbf{k}. \end{aligned}$$

The bond $\beta = (-1 - i, -i, -1 - i, i)$ belongs to the parameter value $t = i$. Because $F_{1,3}(i) \neq 0$, we can compute the local distance as half the vanishing order of

$$Q(F_{1,3}(t)) = (1 + t^2)(t^2 - 2t + 2)$$

at $t = i$, that is, $d_\beta(1, 3) = \frac{1}{2}$. ◇

Example 5. The local distances for the bonds of Example 3 (planar four-bar linkage) are

$$D_{\beta'} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \quad D_{\beta''} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}, \quad D_{\beta'''} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}, \quad (15)$$

where $\beta' = (2 \pm i, \mp i, -1 \pm 2i, \pm i)$, $\beta'' = (\pm i, -2 \pm i, \mp i, 4 \mp i)$, and $\beta''' = (\pm i, \mp i, \pm i, \mp i)$. We introduce a new aspect and discuss the computation of $d_\beta(1, 4)$ for the non-typical bond $\beta = (-i, i, -i, i)$. It corresponds to $t = i$ and the positive square root w in the parametrized equation (13). Because of $f_{1,4} = f_{4,1}^{-1}$ and because inversion is, up to scalar multiplication, equal to conjugation, we clearly have $d_\beta(1, 4) = d_\beta(4, 1)$. The latter local distance is easily computed to be $d_\beta(4, 1) = \frac{1}{2} \text{ord}_\beta Q(t_1 - h_1) = \frac{1}{2}$.

However, when we insert the parametrized equation (13) into the product $F_{1,4} = (t_2 - h_2)(t_3 - h_3)(t_4 - h_4)$, we see that it vanishes at $t = i$. Thus, the parametrization (13) does not give a well-defined map into \mathbb{P}^7 at the bond β and we have to compute the local distance as

$$d_\beta(1, 4) = \frac{1}{2} \text{ord}_\beta Q(F_{1,4}) - \min \text{ord}_\beta(F_{1,4}) = \frac{1}{2} \quad (16)$$

where $\min \text{ord}_\beta(F_{1,4})$ denotes the minimal vanishing order of the coordinates of $F_{1,4}$ at β . This vanishing order enters with multiplicity two in the norm, so that the factor $\frac{1}{2}$ can be omitted. The actual evaluation of (16) by means of the parametrized equation (13) poses no problems. ◇

Below, we present an alternative method for computing local distances using the products $F_{i,j}$ as functions from $N(K)$ to $\mathbb{D}\mathbb{H}$. As a consequence, we are able to derive a couple of interesting properties of the local distance function (Theorem 15).

Lemma 14. *The local distance $d_\beta(i, j)$ can be computed as*

$$d_\beta(i, j) = \sum_{k=i+1}^j b_\beta(k) - v_\beta(i, j) \quad (17)$$

where $v_\beta(i, j) = \min \text{ord}_\beta(F_{i,j})$ is the minimal vanishing order of the coordinates of $F_{i,j}$ at β .

Proof. If $v_\beta(i, j) = 0$, then the product of $F_{k-1,k}$ for $k = i+1, \dots, j$ does not vanish at β , and gives $F_{i,j}$. The primal part of the norm is multiplicative, and this implies the equation. In the general case, the product is equal to $u^m f_{i,j}$ for some local parameter u at β and $m = v_\beta(i, j)$, and this gives precisely the correction stated in the equation. \square

Example 6. We continue Example 4 and compute $d_\beta(1, 3)$ at $\beta = (-1 - i, -i, -1 - i, i)$ also by means of Lemma 14. From the matrices in (14) we read off:

$$b_\beta(2) = d_\beta(1, 2) = \frac{1}{2}, \quad b_\beta(3) = d_\beta(2, 3) = 0.$$

It is easy to verify that $F_{1,3}(i) \neq 0$ so that $\min \text{ord}_\beta(f_{1,3}) = 0$ and

$$d_\beta(1, 3) = b_\beta(2) + b_\beta(3) - v_\beta(1, 3) = \frac{1}{2} + 0 - 0 = \frac{1}{2},$$

as expected. \diamond

Example 7. We continue with Example 5 and compute $d_\beta(1, j)$ at $\beta = (i, -i, i, -i)$ also by means of Lemma 14. From Equation (15) we see that

$$b_\beta(2) = b_\beta(3) = b_\beta(4) = \frac{1}{2}.$$

For computing the local distances, we also need the vanishing orders $v_\beta(1, j)$. Since β belongs to the parameter value $t = i$ in the parametrization (13) with positive sign of the square root, we have to compute the minimum vanishing order of the coordinates of $F_{1,j}(t)$ at $t = i$. We have

$$F_{1,2}(i) \neq 0, \quad F_{1,3}(i) \neq 0, \quad F_{1,4}(i) = 0, \quad \frac{d}{dt} F_{1,4}(i) \neq 0.$$

Hence $v_\beta(1, 2) = v_\beta(1, 3) = 0$, $v_\beta(1, 4) = 1$ and

$$d_\beta(1, 2) = \frac{1}{2} - 0 = \frac{1}{2}, \quad d_\beta(1, 3) = \frac{1}{2} + \frac{1}{2} - 0 = 1, \quad d_\beta(1, 4) = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 1 = \frac{1}{2},$$

as expected. \diamond

From these examples, some properties of local bonds are fairly obvious. We state and prove them formally in

Theorem 15. *For each bond β , the local distance d_β has the following properties:*

a) *The local distance is a pseudometric on $[n]$: For all $i \leq j \leq k \in [n]$ we have*

- $d_\beta(i, i) = 0$,
- $d_\beta(i, j) = d_\beta(j, i)$,
- $d_\beta(i, k) \leq d_\beta(i, j) + d_\beta(j, k)$ (triangle inequality).

b) $d_\beta(i, j) + d_\beta(j, k) + d_\beta(i, k) \in \mathbb{N}$

c) $d_\beta(i - 1, i + 1) = b_\beta(i) + b_\beta(i + 1)$

Proof. a) The first item is true because $d_\beta(i, i)$ is the vanishing order of the empty product whose value is defined to be 1. The second item is true because $d_\beta(j, i)$ is the vanishing order of $Q(f_{j,i})$ at β . It equals $Q(f_{i,j})$ because $f_{j,i}$ is the conjugate of $f_{i,j}$. In order to prove the triangle inequality, we observe that $v_\beta(i, j) + v_\beta(j, k) \leq v_\beta(i, k)$ because the formal product for computing the right-hand side can be factored into the formal products for computing the left-hand side. Thus, by Equation (17), we have

$$\begin{aligned} d_\beta(i, j) + d_\beta(j, k) &= \sum_{l=i+1}^j b_\beta(l) - v_\beta(i, j) + \sum_{l=j+1}^k b_\beta(l) - v_\beta(j, k) \\ &= \sum_{l=i+1}^k b_\beta(l) - v_\beta(i, j) - v_\beta(j, k) \geq \sum_{l=i+1}^k b_\beta(l) - v_\beta(i, k) = d_\beta(i, k). \end{aligned}$$

b) By Equation (17) we have

$$d_\beta(i, j) + d_\beta(j, k) + d_\beta(i, k) = 2 \sum_{l=i+1}^k b_\beta(l) - v_\beta(i, j) - v_\beta(j, k) - v_\beta(i, k).$$

The right-hand side is a sum of integers and the left-hand side is non-negative.

c) is equivalent to $v_\beta(i - 1, i + 1) = 0$, that is, the product $(t_i - h_i)(t_{i+1} - h_{i+1})$ does not vanish at β . Expanding this product and assuming, indirectly, that it does vanish, we get a nontrivial relation with complex coefficients between the vectors $1, h_i, h_{i+1}, h_i h_{i+1}$. Its real or complex part is a nontrivial relation with real coefficients. Under our general assumption that two consecutive revolute axes are never identical, this contradicts Theorem 5.b). \square

3.3 Distances and joint lengths

Now we introduce (non-local) distances and joint lengths and relate them to local distances and joint lengths.

Definition 16. The *distance* $d(i, j)$ is defined as $d(i, j) := \deg(C_{i,j}) \deg(f_{i,j})$, where $\deg(C_{i,j})$ is the degree of the coupler curve as a projective curve in \mathbb{P}^7 and $\deg(f_{i,j})$

is the algebraic degree of the coupler map $f_{i,j}: K \rightarrow C_{i,j}$, that is, the cardinality of a generic pre-image when we consider also complex points of K . Moreover, we write $b(i) := d(i-1, i)$ for $i = 1, \dots, n$, $d(0, 1) = d(n, 1)$ and call the numbers $b(1), \dots, b(n)$ the *joint lengths*.

The definition of $d(i, j)$ as geometric degree times multiplicity suggests to refer to it also as *algebraic degree* of the coupler curve $C_{i,j}$.

It is a good point to clarify some of our terminology. When we speak of a coupler curve, we mean the relative motion between two links. In the Study quadric model of Euclidean displacements this is, indeed, a curve. To us, the degree of a motion is the degree of the corresponding curve on the Study quadric. This differs from the notion of a motion's degree as the degree of a generic trajectory. Twice the degree of the curve on the Study quadric is an upper bound for the trajectory degree.

Since the coupler curve $C_{i-1,i}$ is a straight line (corresponding to the rotation around the axis h_i), $\deg(C_{i-1,i}) = 1$ and $b(i)$ just equals the degree of the map $f_{i-1,i}$. In particular, if $b(i) = 1$, all coupler curves can be parametrized by the revolute angle at h_i (this angle unambiguously determines the linkage configuration).

Theorem 17. a) The distance d is the sum of the local distances: $d(i, j) = \sum_{\beta} d_{\beta}(i, j)$ for all $i, j \in [n]$.
b) The distance d is a pseudometric on $[n]$.
c) For $i \leq j \leq k \in [n]$, $d(i, j) + d(j, k) + d(i, k)$ is a positive even integer.
d) For $i \in [n]$, we have $d(i-1, i+1) = b(i) + b(i+1)$.

Proof. a) For computing $d(i, j)$, we can take any quadratic form that does not vanish on $C_{i,j}$, count the points in $N(K)$ where this form vanishes (counting means with multiplicities), and divide by two. We take Q , the primal part of the norm, as quadratic form. The points where Q vanishes are bonds, and the multiplicity of β is $2d_{\beta}(i, j)$.

b), c) and d) are easy consequences of a) and the corresponding statements in Theorem 15. For statement c), we also need to observe that bonds always come as conjugate pairs, and the local distances for conjugate bonds are equal. \square

The importance of Theorem 17 lies in the fact that it connects local distances, which are part of the bond structure and have an algebraic meaning, with distances (or algebraic degrees), which have a geometric meaning. We collect the distances in the *distance matrix* $D = \sum_{\beta} D_{\beta}$.

Example 8. The distance matrices for the Bennett linkage example, the planar four-bar example, and the Goldberg linkage are

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 2 & 4 & 2 \\ 2 & 0 & 2 & 4 \\ 4 & 2 & 0 & 2 \\ 2 & 4 & 2 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 2 & 3 & 2 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 1 & 0 & 1 & 2 \\ 3 & 2 & 1 & 0 & 1 \\ 2 & 3 & 2 & 1 & 0 \end{pmatrix}, \quad (18)$$

respectively. The first and second matrix are obtained by adding the matrices given in Equation (14) and Equation (15), respectively, and multiplying them by two (because the bonds come in conjugate complex pairs with identical local joint distances). The matrix for the Goldberg linkage was obtained by means of Theorem 21 from the bond diagram in Figure 1.c).

In the Bennett case, neighboring links have a relative motion of degree one (a rotation about their common axes) and opposite links have a relative motion of degree two. In the planar four-bar case, the relative motion of neighboring links is still a rotation but a generic rotation angle occurs twice. Hence, this motion is of degree two. The relative motion of opposite links is of degree four. These well-known facts are confirmed by Equations (14) and (15) in conjunction with Definition 16. \diamond

3.4 Connection numbers and bond diagrams

In this section, we define the *connection number* for two joints and visualize it in *bond diagrams*. These are linkage graphs (with vertices denoting links and edges denoting joints) augmented with additional connections between certain edges. They serve as a pictorial representation for part of the information encoded in the linkage's bond structure. It is possible to directly “read off” certain linkage properties from its bond diagram.

Consider a typical bond β with $t_i^2 + 1 = t_j^2 + 1 = 0$ for $i < j \in [n]$. From the linkage graph we remove the edges labeled h_i and h_j , thus producing two unconnected chain graphs. Then $d_\beta(k, l) = 0$ if the vertices labeled o_k and o_l are in the same component and $d_\beta(k, l) = d_\beta(i - 1, i)$ if they are in different components. We say that the *connection number* $k_\beta(i, j)$ for this typical bond is equal to $2d_\beta(i - 1, i)$ or that *the bond β connects h_i and h_j with multiplicity $2d_\beta(i - 1, i)$* . For the typical bonds in our examples, we always have $k_\beta(i, j) = 1$. Pictorially, a typical bond β cuts the link diagram into two parts, which are separated by a fixed distance, generically $\frac{1}{2}$. The same holds true for the conjugate bond $\bar{\beta}$. Both bonds together account for the total connection number of $2d_\beta(i - 1, i)$.

Definition 18. A bond β is called *elementary*, if $\sum_{i=1}^n b_\beta/2 = 1$.

Every elementary bond is typical but a typical bond need not be elementary. The typical bonds in our examples are all elementary. For a linkage with only elementary bonds, the number $k(i, j)$ of bonds connecting h_i and h_j equals

$$k(i, j) = d(i, j) + d(i - 1, j - 1) - d(i, j - 1) - d(i - 1, j). \quad (19)$$

Indeed, for an elementary bond β we have

$$d_\beta(i, j) + d_\beta(i - 1, j - 1) - d_\beta(i, j - 1) - d_\beta(i - 1, j) = \begin{cases} 1 & \text{if } \beta \text{ connects } h_i \text{ and } h_j, \\ 0 & \text{else.} \end{cases}$$

By Theorem 17.a), the right-hand side of (19) really counts the bonds connecting h_i and h_j . These observations for elementary bonds motivate the following definition for the general setting.

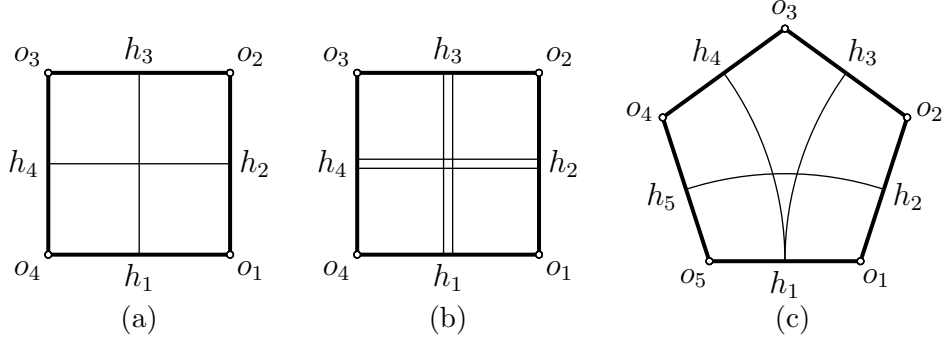


Figure 1: Bond diagrams for the Bennett linkage (a), spherical and planar four-bar (b), and Goldberg linkage (c)

Definition 19. For a closed linkage $L = (h_1, \dots, h_n)$ with bond β and $i < j \in [n]$, the *connection number* $k_\beta(i, j)$ at β is defined as

$$k_\beta(i, j) = d_\beta(i, j) + d_\beta(i - 1, j - 1) - d_\beta(i, j - 1) - d_\beta(i - 1, j). \quad (20)$$

We also say that *the bond β connects the joints h_i and h_j with multiplicity $k_\beta(i, j)$* .

Lemma 20. *The connection number $k_\beta(i, j)$ is an integer.*

Proof. By (17) and (20), we have

$$k_\beta(i, j) = v_\beta(i, j - 1) + v_\beta(i - 1, j) - v_\beta(i, j) - v_\beta(i - 1, j - 1).$$

This is a sum of integers. □

We visualize a bond and its connection number by *bond diagrams*. These are obtained by drawing $k_\beta(i, j)$ connecting lines between the edges h_i and h_j for each set $\{\beta, \bar{\beta}\}$ of conjugate complex bonds. Since we cannot exclude that $k_\beta(i, j) < 0$, we visualize negative connection numbers by drawing the appropriate number of dashed connecting lines (because the dash resembles a “minus” sign). No example in this article has negative connection number. Actually, the authors do not know if closed 6R linkages may or may not have bonds with negative connection numbers.

Example 9. The bond diagrams for our prototype examples, the Bennett linkage and the planar four-bar linkage, are depicted in Figure 1.a) and b). The elementary bonds with $t_i^2 + 1 = t_j^2 + 1 = 0$ connect only h_i and h_j with connection multiplicity one. The non-typical bond of the planar four-bar example connects h_1 with h_3 and h_2 with h_4 , both with connection multiplicity one. Its local distance matrix is sum of the elementary bonds’ distance matrices. We remark that Figure 1.b) also gives the bond diagram for the spherical four-bar linkage of Example 2. Intuitively, two elementary bonds of the spherical four-bar coincide in the planar four-bar. ◇

The algebraic degrees of relative coupler motions $C_{i,j}$ are the entries $d(i,j)$ of the linkage's distance matrix $D = \sum_{\beta} D_{\beta}$. These entries can also be read off directly from the bond diagram which gives us the connection numbers. The following theorem describes how to do this. It is, essentially, a graphical method to invert the linear map δ defined by Equation (20).

Theorem 21. *The algebraic degree of the coupler curve $C_{i,j}$ can be read off from the bond diagram as follows: Cut the bond diagram at the vertices o_i and o_j to obtain two chain graphs with endpoints o_i and o_j ; the degree of $C_{i,j}$ is the sum of all connections that are drawn between these two components (dashed connections counted negatively).*

Proof. Let β be a bond. For any two different links o_i and o_j , the number of connections between the two subchains obtained by cutting the bond diagram at o_i and o_j belonging to β is equal to

$$a_{\beta}(i,j) := \sum_{r=i+1}^j \sum_{s=j+1}^i k_{\beta}(r,s). \quad (21)$$

We identify the space of symmetric $n \times n$ matrices with zero diagonal with \mathbb{R}^N , $N = \binom{n}{2}$ and denote by K_{β} and A_{β} the matrices with respective entries $k_{\beta}(i,j)$ and $a_{\beta}(i,j)$. Equations (20) and (21) define two linear maps $f, g: \mathbb{R}^N \rightarrow \mathbb{R}^N$, $f(D_{\beta}) = K_{\beta}$ and $g(K_{\beta}) = A_{\beta}$. We claim that $f \circ g$ is twice the identity. Indeed, a summand $k_{\beta}(r,s)$ in

$$\sum_{r=i+1}^j \sum_{s=j+1}^i k_{\beta}(r,s) + \sum_{r=i}^{j-1} \sum_{s=j}^{i-1} k_{\beta}(r,s) - \sum_{r=i+1}^{j-1} \sum_{s=j}^i k_{\beta}(r,s) - \sum_{r=i}^{j-1} \sum_{s=j+1}^{i-1} k_{\beta}(r,s)$$

occurs four times with signs $+, +, -, -$ if $\{r,s\} \cap \{i,j\} = \emptyset$, twice with signs $+, -$ if $\{r,s\}$ and $\{i,j\}$ have one element in common, and twice with signs $+, +$ if $\{r,s\} = \{i,j\}$.

Since f, g are linear maps between finite-dimensional vectorspaces, it follows that $g \circ f$ is also twice the identity. Therefore $d_{\beta}(i,j) = \frac{a_{\beta}(i,j)}{2}$ for all pairs i,j such that $i \neq j$. By summing over all bonds, we get the theorem. \square

Example 10. We illustrate the procedure for computing the distances (or coupler curve degrees) in Figure 2. In order to determine the degree of the coupler curve $C_{3,5}$, we cut the bond diagram along the line through o_3 and o_5 and count the connections between the two chain graphs. There are precisely two of them, one connecting h_1 with h_4 and one connecting h_2 with h_5 . Thus, the algebraic degree $d(3,5)$ of $C_{3,5}$ is two. The reader is invited to compute the complete data of Equation (18) by means of the bond-diagrams in Figure 1. \diamond

In the beginning, when we learned the properties of bonds mostly from observation, the majority of linkages we studied had only simple bonds. It occurred to us that these special points on the configuration curve somehow mysteriously connects two of the n joints, which are not joined by a link. This is the reason for the name “bond”. We emphasize that it should not be confused with the already established concept of a “kinematic bond” [1, Chapter 5].

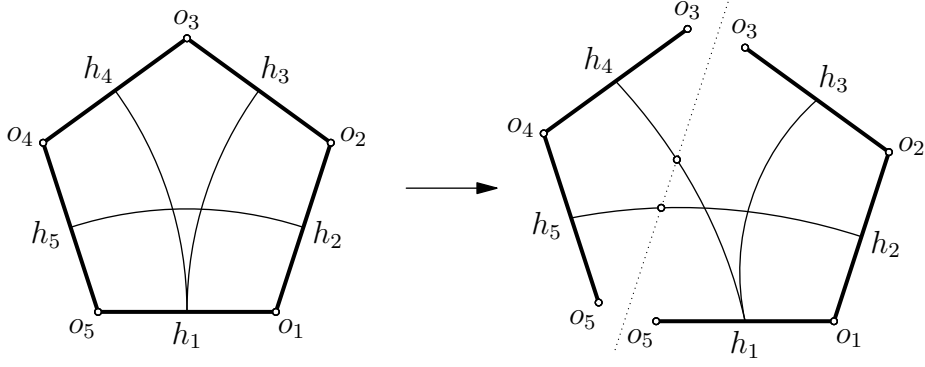


Figure 2: Computing the degree of coupler curves by counting connections in the bond-diagram

3.5 More properties of bonds

We briefly state a few additional properties of bonds that follow immediately from our considerations so far or can easily be shown. We talk about the bonds of the linkage $L = (h_1, \dots, h_n)$.

Corollary 22. *If $d(1, 4) < d(1, 2) + d(2, 3) + d(3, 4)$, then $l_{234} \leq 6$.*

Proof. There must exist at least one bond such that $d_\beta(1, 4) < d_\beta(1, 2) + d_\beta(2, 3) + d_\beta(3, 4)$. For this bond, call it β , we have $v_\beta(1, 4) > 0$ by Lemma 14. Let t_2, t_3, t_4 be the second, third, and fourth coordinate of β , respectively. Since $v_\beta(1, 4) > 0$, the formal product of the corresponding rotations vanishes at β , i.e. $(t_2 - h_2)(t_3 - h_3)(t_4 - h_4) = 0$. Expanding this product, we get a nontrivial relation with complex coefficients between the vectors $1, h_2, h_3, h_4, h_2h_3, h_2h_4, h_3h_4, h_2h_3h_4$. Either its real or its complex part is a nontrivial relation with real coefficients. So l_{234} cannot be eight. By Theorem 5, it cannot be larger than six. \square

Corollary 23. *If a bond β connects h_i with h_{i+2} , the axes of h_i, h_{i+1} and h_{i+2} are concurrent or satisfy the Bennett conditions, compare Theorem 5.d).*

Proof. Without loss of generality, we assume $i = 2$. The connection number $k_\beta(i, i + 2)$ is positive, that is, $d_\beta(2, 4) + d_\beta(1, 3) - d_\beta(2, 3) - d_\beta(1, 4) > 0$. Using Theorem 17.d), we find

$$\begin{aligned} d_\beta(1, 4) &< d_\beta(2, 4) + d_\beta(1, 3) - d_\beta(2, 3) \\ &= d_\beta(2, 3) + d_\beta(3, 4) + d_\beta(1, 2) + d_\beta(2, 3) - d_\beta(2, 3) \\ &= d_\beta(1, 2) + d_\beta(2, 3) + d_\beta(3, 4). \end{aligned}$$

By Corollary 22, this implies $l_{234} \leq 6$ and the claim follows from Theorem 5. \square

Corollary 24. *If a joint h_i is connected with multiplicity one to exactly one other joint, then the configuration curve can be parametrized by t_i , or, equivalently, by the rotation angle at h_i .*

Proof. The assumption is equivalent to $b(i) = 1$. If this is true, then the coupler map $f_{i-1,i}$ is birational, and its inverse is the desired parametrization. \square

Corollary 25. *If two joints h_i, h_j of length $d(i, j) = 1$ are connected with multiplicity one to each other and they are not connected to other joints, then $t_i = \pm t_j$ holds for all points of the configuration curve.*

Proof. This follows from Corollary 24 and the fact that the configuration curve can be parametrized birationally by t_i or by t_j . Hence there is a projective equivalence relating t_i and t_j . This equivalence fixes ∞ and takes the zeroes of $t_i^2 + 1$ to the zeroes of $t_j^2 + 1$. This already implies $t_j = \pm t_i$. \square

3.6 More examples

In this subsection we present three more examples of overconstrained 6R linkages and their bond diagrams. Apparently, the linkages in Examples 11 and 12 are new.

Example 11. We use the method of factorizing motion polynomials [7, 8] to construct a 6R linkage as follows. First, we choose two arbitrary rotation polynomials h_1, h_2 with non-concurrent axes, say $h_1 = \mathbf{i}$ and $h_2 = \epsilon \mathbf{i} + \mathbf{j}$. Then we choose a random linear combination of 1 and h_2 , say $h'_2 = 1 + h_2$, and factor the quadratic motion polynomial as $P(t) = (t - h_1)(t - h'_2)$. We compute a second factorization $P(t) = (t - 1 - g_1)(t - g_2)$. Then we choose another random linear combination of 1 and h_2 , say $h''_2 = 2 + h_2$, and factor the motion polynomial $Q(t) = (t - g_2)(t - h''_2)$. We get a second factorization $Q(t) = (t - 2 - h_4)(t - h_3)$. Next we choose a random linear combination of 1 and h_4 , say $h'_4 = 3 + h_4$, and we factor the motion polynomial $R(t) = (t - 1 - g_1)(t - 3 - h_4)$. We get a second factorization $R(t) = (t - 3 - h_6)(t - 1 - h_5)$. We obtain a six-bar linkage $L = (h_1, h_2, h_3, h_4, h_5, h_6)$ with configuration curve

$$(t_1, t_2, t_3, t_4, t_5, t_6) = \left(t, \frac{t^2 - 3t + 1}{2t - 3}, -t, t^2 - 5t + 7, -t + 1, -t + 3 \right).$$

The bonds are

$$\begin{aligned} & (3 \pm \mathbf{i}, \frac{6}{13} \pm \frac{9}{13}\mathbf{i}, -3 \mp \mathbf{i}, \pm \mathbf{i}, -2 \mp \mathbf{i}, \mp \mathbf{i}), \\ & (\pm \mathbf{i}, -\frac{6}{13} \pm \frac{9}{13}\mathbf{i}, \mp \mathbf{i}, 6 \mp 5\mathbf{i}, 1 \mp \mathbf{i}, 3 \mp \mathbf{i}), \\ & (2 \pm \mathbf{i}, \pm \mathbf{i}, -2 \mp \mathbf{i}, \mp \mathbf{i}, -1 \mp \mathbf{i}, 1 \mp \mathbf{i}), \\ & (1 \pm \mathbf{i}, \pm \mathbf{i}, -1 \mp \mathbf{i}, 2 \mp 3\mathbf{i}, \mp \mathbf{i}, 2 \mp \mathbf{i}). \end{aligned}$$

We have four pairs of conjugate complex bonds. All of them are elementary, the bond diagram is given in Figure 4.a).

The reason why we think that this concrete linkage is new is the following. For $i = 1, \dots, 6$, let a_i be the distance of the lines l_i, l_{i+1} , let α_i be the angle between the lines l_i, l_{i+1} , and let s_i be the distance between the two points on l_i which are in shortest distance to l_{i-1} and l_{i+1} . For almost all examples we found in the literature, there is at least one equality between the a_i , or at least one angle α_i which is zero or a right angle, or an index i such that $s_i = s_{i+1} = s_{i+2} = 0$. There are only two exceptions, namely

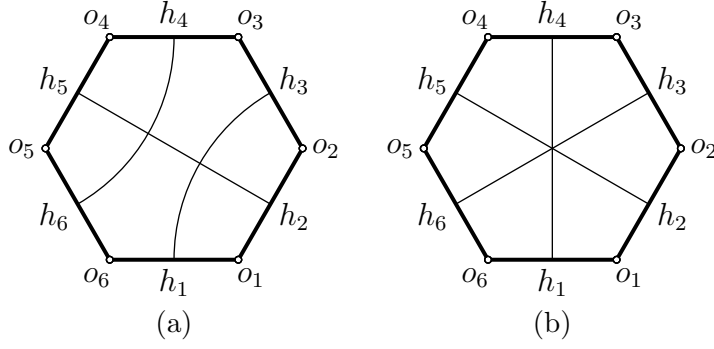


Figure 3: Bond diagrams for Waldron's double Bennett linkage and for the linkage of [7, 8]

Waldron's double Bennett (see [4]) and the linkage of [7, 8]. In our example, the numbers a_i , $i = 1, \dots, 6$, are pairwise distinct, the angles are neither zero nor right angles, and the only vanishing offsets are $s_2 = s_3 = s_5 = 0$. Hence our example is not a special case of a known family, except possibly the double Bennett and the linkages of [7, 8]. But these two families have a different bond structure as shown in Figure 3. So, our linkage is also not a special case of these linkages. \diamond

Example 12. Starting from h_1, \dots, h_6 as in the example above, we factor the motion polynomial $S(t) = (t + h_1)(t - 3 - h_6)$ and get a second factorization $S(t) = (t - 3 - h'_1)(t - h'_6)$. The six-bar linkage $L = (h'_1, h_2, h_3, h_4, h_5, h'_6)$ has configuration curve

$$(t_1, t_2, t_3, t_4, t_5, t_6) = (3 + t, -\frac{t^2 + 3t + 1}{2t + 3}, t, t^2 + 5t + 7, t + 1, t).$$

The bonds are

$$\begin{aligned} &(\mp i, \frac{6}{13} \pm \frac{9}{13}i, -3 \mp i, \pm i, -2 \mp i, -3 \mp i), \\ &(3 \mp i, -\frac{6}{13} \pm \frac{9}{13}i, \mp i, 6 \mp 5i, 1 \mp i, \mp i), \\ &(1 \mp i, \pm i, -2 \mp i, \mp i, -1 \mp i, -2 \mp i), \\ &(2 \mp i, \pm i, -1 \mp i, 2 \mp 3i, \mp i, -1 \mp i), \end{aligned}$$

the corresponding bond diagram is shown in Figure 4.b).

The reason why we think that this example is new is the same as for Example 12, except that now we have only one vanishing offset $s_3 = 0$ (so we would not need to worry about the double Bennett linkage). \diamond

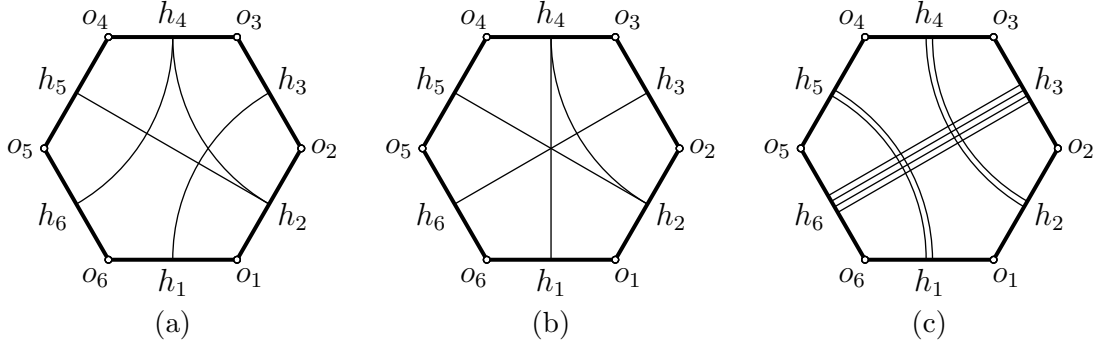


Figure 4: Bond diagrams for the linkages of Examples 11, 12, and 13

Example 13 (Bricard's plane symmetric linkage). We set

$$\begin{aligned}
 h_1 &= \frac{4}{9}\mathbf{i} - \frac{17}{27}\epsilon\mathbf{i} - \frac{7}{9}\mathbf{j} - \frac{4}{27}\epsilon\mathbf{j} + \frac{4}{9}\mathbf{k} + \frac{10}{27}\epsilon\mathbf{k}, \\
 h_2 &= \frac{3}{7}\mathbf{i} + \frac{32}{49}\epsilon\mathbf{i} + \frac{2}{7}\mathbf{j} + \frac{12}{49}\epsilon\mathbf{j} + \frac{6}{7}\mathbf{k} - \frac{20}{49}\epsilon\mathbf{k}, \\
 h_5 &= \frac{4}{9}\mathbf{i} + \frac{17}{27}\epsilon\mathbf{i} + \frac{7}{9}\mathbf{j} - \frac{4}{27}\epsilon\mathbf{j} - \frac{4}{9}\mathbf{k} + \frac{10}{27}\epsilon\mathbf{k}, \\
 h_4 &= \frac{3}{7}\mathbf{i} - \frac{32}{49}\epsilon\mathbf{i} - \frac{2}{7}\mathbf{j} + \frac{12}{49}\epsilon\mathbf{j} - \frac{6}{7}\mathbf{k} - \frac{20}{49}\epsilon\mathbf{k}, \\
 h_3 &= \mathbf{k}, \quad h_6 = \mathbf{j}.
 \end{aligned}$$

It can be seen that the axes of h_3, h_6 lie in a plane, and the axes of h_1, h_5 and h_2, h_4 , respectively, are symmetric with respect to this plane. Thus, we have an example of Bricard's plane symmetric linkage [2], [4, pp. 91–92]. The configuration curve has genus one, hence it is not parametrizable by polynomials. For the whole configuration curve, we have $t_1 = -t_5$ and $t_2 = -t_4$. One observes that the bonds follow the following pattern:

$$(\pm\mathbf{i}, *, *, *, \mp\mathbf{i}, *), \quad (*, \pm\mathbf{i}, *, \mp\mathbf{i}, *, *), \quad (*, *, \pm\mathbf{i}, *, *, \pm\mathbf{i}), \quad (*, *, \pm\mathbf{i}, *, *, \mp\mathbf{i})$$

(the $*$ signs denote complex numbers, all different, with real and imaginary part different from zero). The bond diagram is shown in Figure 4.c). \diamond

4 Classification of closed 5R chains

As an application of bond theory, we give a proof of Karger's classification of overconstrained closed 5R linkages [11]. The main statement is that any non-trivial linkage of this type is a Goldberg linkage. We also compute the degree of the coupler motions of Goldberg's linkage.

In the following we suppose that the linkage $L = (h_1, \dots, h_5)$ is a closed 5R chain with mobility one, which is neither planar nor spherical, i.e. not all five axes are parallel or

meet in a point. We also assume that any two consecutive axes are distinct, and that no coupler map is constant (for instance L is not a 4R linkage plus one fixed link). A 5R linkage fulfilling these conditions is called a non-degenerate 5R linkage.

Lemma 26. *All coupling dimensions $l_{i,\dots,j}$ in a non-degenerate 5R linkage are greater than four.*

Proof. Assume indirectly that there exists a coupling dimension which is four, say $l_{123} = 4$. Then it follows from Theorem 5.c) that the axes of h_1, h_2, h_3 intersect in a common point O , possibly at infinity. The coupler curve C_{54} contains only rotations around axes through O . On the other hand, L_{54} contains only either rotations around h_4 and h_5 (if these two axes are skew) or rotations around axes through the common intersection point O' (possibly at infinity), if h_4 and h_5 are not skew. Hence $O = O'$ and L is a planar or spherical linkage. \square

Lemma 27. *For $i \in [n]$ the coupler map $f_{i,i+2}$ is injective.*

Proof. Without loss of generality, we assume $i = 5$. The parametrization $(\mathbb{P}^1)^2 \rightarrow X_{12}$, $(t_1, t_2) \mapsto (t_1 - h_1)(t_2 - h_2)$ is injective. Similarly, it follows from $l_{543} > 4$ that the parametrization $(\mathbb{P}^1)^3 \rightarrow X_{543}$, $(t_5, t_4, t_3) \mapsto (t_5 - h_5)(t_4 - h_4)(t_3 - h_3)$ is injective. Hence there can be at most one configuration $(t_1, t_2, t_3, t_4, t_5)$ that maps into some point in the intersection $X_{12} \cap X_{541}$. \square

Lemma 28. *Let h_1, \dots, h_6 be six half-turns such that $L_{123} = L_{456} =: L$ and $\dim(L) = 6$. Then $h_1 = \pm h_4$ and $h_3 = \pm h_6$.*

Proof. Let $A \subset \mathbb{DH}$ be the set of all elements a such that L is closed under multiplication with a from the left. Then A is a subalgebra, we have $h_1 \in A$ and $h_4 \in A$, and $A \subset L$ because $1 \in L$. Assume, indirectly, that $h_1 \neq \pm h_4$. The only proper subalgebras of \mathbb{DH} containing two different rotations are conjugate to $\text{SO}_3 = \mathbb{H}$ (rotations about one fixed point) or to $\text{SE}_2 = \langle 1, \mathbf{i}, \epsilon \mathbf{j}, \epsilon \mathbf{k} \rangle$ (rotations about axes parallel to a fixed direction and translation orthogonal to this direction; angled brackets denote linear span). The former does not act by left-multiplication on a module of real dimension 6. The latter acts exactly on one submodule of \mathbb{DH} containing 1, namely $\langle 1, \mathbf{i}, \epsilon \mathbf{j}, \epsilon \mathbf{k}, \epsilon, \epsilon \mathbf{i} \rangle$, which must then be L (up to conjugation). But all rotations in this submodule are contained in A , hence $h_1, h_2, h_3 \in A$ and $A = L$, which is a contradiction. \square

Lemma 29. *If $l_{123} = l_{345} = 6$, then $b(1) = b(2) = b(4) = b(5) = 1$ and $b(3) = 2$.*

Proof. Let $L := L_{123} \cap L_{543}$. Its dimension is even, because L is an L_3 -right vectorspace. By Lemma 28 the spaces L_{123} and L_{543} are different. Hence $\dim(L) \leq 4$. On the other hand, $\dim(L) \geq l_{123} + l_{654} - 8$. Hence, we have $\dim(L) = 4$.

First we prove that $d(3, 5) = 2$. By Theorem 17.d we have $d(3, 5) = b(4) + b(5) \geq 1 + 1 = 2$ ($b(i) > 0$ because all joints move). Assume, indirectly, that $d(3, 5) \geq 3$. Then C_{35} is a curve of degree at least three, because $\deg f_{3,5} = 1$ by Lemma 27. On the other hand, the ideal of C_{35} is generated by linear and quadratic equations, because $C_{35} = X_{123} \cap X_{54}$ and the ideals of X_{123} and of X_{54} are generated by linear and quadratic

equations. Hence, C_{35} is not a plane curve, because otherwise the degree of C_{35} would be at most two. Because of $C_{35} \subset L' := L_{54} \cap L_{123}$, this implies $\dim L' = 4$ and, thus, $L = L'$. But then $L_{54} \subseteq L_{123}$. If we multiply both sides with h_3 from the right, we get $L_{543} \subseteq L_{123}$. Both spaces have the same dimension, hence $L_{123} = L_{543}$ – a contradiction. This proves $d(3, 5) = 2$ and also $b(4) = b(5) = 1$. Applying the same argument for the linkage $(h_5, h_4, h_3, h_2, h_1)$, we get $b(1) = b(2) = 1$.

It remains to be shown that $b(3) = 2$. By the triangle inequality, $b(3) = d(2, 3) \leq d(2, 5) + d(3, 5) = b(1) + b(2) + b(4) + b(5) = 4$. By Theorem 17.c, $d(2, 5) + d(2, 3) + d(3, 5)$ is even, hence the bond length $b(3)$ is even. Clearly, $0 < b(3) \leq 4$. If $b(3) = 4$, then $d(1, 3) = b(2) + b(3) = 5$, contradicting the triangle inequality $d(1, 3) \leq d(1, 5) + d(3, 5) = b(1) + b(4) + b(5) = 3$. Thus, $b(3) = 2$ and the proof is finished. \square

Lemma 30. *Let L be a non-degenerate 5R linkage. Then exactly one of its joint lengths is equal to two, and all others are equal to one.*

Proof. By Lemma 26, the numbers $l_{i,i+1,i+2}$ can only be 6 or 8 for $i = 1, \dots, 5$ (the indices are labeled modulo 5). Because 5 is an odd number, there exists an index i such that $l_{i-2,i-1,i} = l_{i,i+1,i+2}$. Without loss of generality, we may assume $i = 3$. We distinguish two cases.

Case 1: $l_{123} = l_{345} = 8$. By Theorem 17.d and Corollary 22, $l_{123} = 8$ implies $d(5, 3) = d(5, 1) + d(1, 2) + d(2, 3) = b(1) + b(2) + b(3) = b(4) + b(5)$. Similarly, $l_{345} = 8$ implies $b(3) + b(4) + b(5) = b(1) + b(2)$. Hence $b(3) = 0$, a contradiction.

Case 2: $l_{123} = l_{345} = 6$. Then Lemma 29 applies. \square

We are already in a position to state a new result on overconstrained 5R linkages:

Theorem 31. *The coupler motions of a non-degenerate 5R linkage can be parametrized by four of the five joint angles. Its coupler curves (that is, the relative motions as curves on the Study quadric) are plane conics and twisted cubics.*

Proof. The coupler curves can be parametrized by the angles at all four joints of length one. The coupler curves $C_{i-1,i+2}$ have degree $d(i, i+2) = b(i+1) + b(i+2)$ for $i = 1, \dots, 5$ (with cyclic numbering of indices), and this is two or three. Since the ideals of the coupler curves are generated by linear and quadratic forms, they can only be plane conics or twisted cubics. \square

The main result of this section is

Theorem 32. *Every non-degenerate 5R linkage is a Goldberg linkage.*

Proof. Denote the linkage by $L = (h_1, h_2, h_3, h_4, h_5)$. By Lemma 30, there is one joint, say h_3 , of length two, that is $b(3) = 2$. The coupler curve C_{14} is a twisted cubic, in particular it is a rational curve of degree three. We fix a cubic parametrization $\phi : t \mapsto P(t)$ of degree three and apply the synthesis method of [7, 8] for synthesizing open 3R chains that are parametrized linearly and that produce the motion ϕ .

By general results of [7, 8], the relative motion C_{14} admits parametrizations

$$(t - h_3'')(t - h_3')(t - h_2'), \quad (t - h_4')(t - h_5')(t - h_1'), \quad (t - h_3'')(t - h_6')(t - h_1')$$

with $h_i' \in L_i$ for $i = 1, \dots, 6$ and $h_3'' \in L_3$ such that (h_1', h_2', h_3', h_6') and $(h_3'', h_4', h_5', h_6')$ is a Bennett quadruple. The original 5R linkage can be constructed by composition of these two Bennett linkages, with the common axes h_3 , h_6 , and subsequent removal of the joint at h_6 . This is exactly Goldberg's construction [5, 14]. \square

5 Conclusion

This article featured a rigorous introduction of bonds, connection numbers and bond multiplicities. Bond theory is a new tool for analyzing overconstrained linkages and one can hope to gain new insight into their behavior. In order to demonstrate the usefulness of bond theory, we gave a new proof of Karger's classification theorem for overconstrained 5R chains. In a next step, we plan to work out the bond structure for overconstrained 6R chains, both known and new. Some examples have already been given in this paper.

Acknowledgements

We would like to thank Zijia Li (RICAM Linz) for helping us with computations of examples. This research was supported by the Austrian Science Fund (FWF): I 408-N13 and DK W 1214-N15.

References

- [1] J. Angeles. *Rational Kinematics*, volume 34 of *Springer Tracts in Natural Philosophy*. Springer, New York, 1989.
- [2] J. E. Baker. An analysis of Bricard linkages. *Mech. Mach. Th.*, 15(4):267–286, 1980.
- [3] W. Blaschke. *Kinematik und Quaternionen*. VEB Deutscher Verlag der Wissenschaften, Berline, 1960.
- [4] P. Dietmaier. *Simply overconstrained mechanisms with rotational joints (habilitation thesis)*. Univ. Graz, 1995.
- [5] M. Goldberg. New five-bar and six-bar linkages in three dimensions. *Trans. ASME*, 65:649–656, 1943.
- [6] K. Hao. Dual number method, rank of a screw system and generation of Lie sub-algebras. *Mech. Mach. Theory*, 33(7):1063–1084, 1998.
- [7] G. Hegedüs, J. Schicho, and H.-P. Schröcker. Construction of overconstrained linkages by factorization of rational motions. In Jadran Lenarčič and Manfred Husty,

- editors, *Latest Advances in Robot Kinematics*, pages 213–220. Springer, 2012. doi: 10.1007/978-94-007-4620-6_27.
- [8] G. Hegedüs, J. Schicho, and H.-P. Schröcker. Factorization of rational curves in the Study quadric and revolute linkages. Submitted for publication, 2012. URL <http://geometrie.uibk.ac.at/schroecker/qf/>.
 - [9] G. Hegedüs, J. Schicho, and H.-P. Schröcker. Bond theory and closed 5R linkages. In Jadran Lenarčič and Manfred Husty, editors, *Latest Advances in Robot Kinematics*, pages 221–228. Springer, 2012. doi: 10.1007/978-94-007-4620-6_28.
 - [10] M. Husty and H.-P. Schröcker. Algebraic geometry and kinematics. In I. Z. Emiris, F. Sottile, and Th. Theobald, editors, *Nonlinear Computational Geometry*, volume 151 of *The IMA Volumes in Mathematics and its Applications*, chapter Algebraic Geometry and Kinematics, pages 85–107. Springer, 2010.
 - [11] A. Karger. Classification of 5R closed kinematic chains with self mobility. *Mech. Mach. Th.*, pages 213–222, 1998.
 - [12] J. Selig. *Geometric Fundamentals of Robotics*. Monographs in Computer Science. Springer, 2 edition, 2005.
 - [13] I. R. Shafarevich. *Basic Algebraic Geometry*. Springer, 1974.
 - [14] K. Wohlhart. Merging two general Goldberg 5R linkages to obtain a new 6R space mechanism. *Mech. Mach. Th.*, 26:659–668, 1991.